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Path integral for a relativistic Aharonov–Bohm–Coulomb system

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Abstract. The path integral for the relativistic spinless Aharonov–Bohm–Coulomb system is performed. The energy spectra and wavefunctions are extracted from the resulting amplitude.

1. Introduction

With the help of Duru and Kleinert's path-dependent time transformation [1] the list of solvable path integrals has been extended to essentially all potential problems which possess a solvable Schrödinger equation [2, 3]. Only recently has the technique been extended to relativistic potential problems [4], followed by two applications [6–9]. Here we would like to add a further application by solving the path integral of a relativistic particle in two dimensions in the presence of an infinitely thin Aharonov–Bohm magnetic field along the z-axis [10] and a $1/r$-Coulomb potential (ABC system). This may be relevant for understanding the behaviour of relativistic charged anyons which are restricted to a plane but whose Coulomb field extends into three dimensions [2, 11].

This paper is organized as follow. In section 2, we calculate the path integral of the relativistic ABC potential problem. The energy spectra and wavefunctions are extracted from the resulting amplitude. Our conclusions are summarized in section 3.

2. Path integral for a relativistic Aharonov–Bohn–Coulomb system

Let us first consider a point particle of mass M moving at a relativistic velocity in a $(D+1)$-dimensional Minkowski space with a given electromagnetic field. By using $t = -\mathrm{i}\tau = -\mathrm{i}x^4/c$, the path integral representation of the fixed-energy amplitude (Green function) is conveniently formulated in a $(D+1)$-Euclidean spacetime with the Euclidean metric,

$$(g_{\mu\nu}) = \mathrm{diag}\,(1, \ldots, 1, c^2) \tag{1}$$

and is given by [4, 5]

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc}\int_0^\infty \mathrm{d}L \int D\rho\,\Phi[\rho] \int D^D x\, \mathrm{e}^{-A_E/\hbar} \tag{2}$$

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with the action

$$A_E = \int_{\tau_a}^{\tau_b} \mathrm{d}\tau \left[\frac{M}{2\rho(\tau)} \boldsymbol{x}'^2(\tau) - \mathrm{i}\frac{e}{c} \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{x}'(\tau) - \rho(\tau) \frac{(E - V(x))^2}{2Mc^2} + \rho(\tau) \frac{Mc^2}{2} \right] \tag{3}$$

where L is defined by

$$L = \int_{\tau_a}^{\tau_b} \mathrm{d}\tau \; \rho(\tau) \tag{4}$$

in which $\rho(\tau)$ is an arbitrary dimensionless fluctuating scale variable, and $\Phi[\rho]$ is some convenient gauge-fixing functional, such as $\Phi[\rho] = \delta[\rho - 1]$, to fix the value of $\rho(\tau)$ to unity [4, 6, 7]. $\hbar/Mc$ is the well known Compton wavelength of a particle of mass M, $\boldsymbol{A}(\boldsymbol{x})$ is the vector potential, $V(\boldsymbol{x})$ is the scalar potential, E is the system energy, and $\boldsymbol{x}$ is the spatial part of the $(D+1)$ vector $x = (\boldsymbol{x}, \tau)$. This path integral forms the basis for studying relativistic potential problems.

For the ABC system under consideration, the scalar potential is

$$V(r) = -e^2/r \tag{5}$$

and the vector potential reads

$$\boldsymbol{A}(\boldsymbol{x}) = 2g \frac{-x_2 \hat{e}_1 + x_1 \hat{e}_2}{x_2^2 + x_1^2} \tag{6}$$

where e is the charge and $\hat{e}_{1,2}$ stand for the unit vector along the x, y axis, respectively. For convenience, we introduce the azimuthal angle around the tube:

$$\theta(\boldsymbol{x}) = \arctan(x_2/x_1). \tag{7}$$

The components of the vector potential can be, therefore, expressed as

$$A_i = 2g\partial_i\theta. \tag{8}$$

The associated magnetic field lines are confined to an infinitely thin tube along the z-axis:

$$B_3 = 2g\epsilon_{3jk}\partial_j\partial_k\theta = 2g2\pi\delta^{(2)}(\boldsymbol{x}_\perp) \tag{9}$$

where $\boldsymbol{x}_\perp$ is the transverse vector $\boldsymbol{x}_\perp \equiv (x_1, x_2)$.

To obtain a tractable path integral for the potential V, we have to regularize it via a so-called f-transformation [2, 6], which exchanges the path parameter τ by a new one s:

$$\mathrm{d}\tau = \mathrm{d}s \; f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1}) \tag{10}$$

where $f_l(\boldsymbol{x})$ and $f_r(\boldsymbol{x})$ are invertible functions whose product is positive. The freedom in choosing $f_{l,r}$ amounts to an invariance under path-dependent reparametrizations of the path parameter τ in the fixed-energy amplitude of equation (2). By this transformation, the $(D+1)$-dimensional relativistic fixed-energy amplitude for arbitrary time-independent potential turns into the lattice form [2, 6]

$$\begin{aligned} G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) \approx \frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S \prod_{n=1}^{N+1} \left[\int \mathrm{d}\rho_n \, \Phi(\rho_n) \right] \frac{f_l(\boldsymbol{x}_a) f_r(\boldsymbol{x}_b)}{[2\pi\hbar\epsilon_b^s \rho_b f_l(\boldsymbol{x}_b) f_r(\boldsymbol{x}_a)/M]^{D/2}} \\ \times \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}^D x_n}{[2\pi\hbar\epsilon_n^s \rho_n f(\boldsymbol{x}_n)/M]^{D/2}} \right] \exp\left\{ -\frac{1}{\hbar} A^N \right\} \end{aligned} \tag{11}$$

with the s-sliced action

$$\begin{aligned} A^N = \sum_{n=1}^{N+1} \bigg[\frac{M(\boldsymbol{x}_n - \boldsymbol{x}_{n-1})^2}{2\epsilon_n^s \rho_n f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1})} - \mathrm{i}\frac{e}{c} \boldsymbol{A}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \\ -\epsilon_n^s \rho_n f_l(\boldsymbol{x}_n) f_r(\boldsymbol{x}_{n-1}) \frac{(E - V_n)^2}{2Mc^2} + \epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\boldsymbol{x}_{n-1}) \frac{Mc^2}{2} \bigg] \end{aligned} \tag{12}$$

where the sign $\approx$ in equation (11) becomes an equality for $N \to \infty$. A family of functions which regulates the ABC system is

$$f_l(\boldsymbol{x}) = f(\boldsymbol{x})^{1-\lambda} \qquad f_r(\boldsymbol{x}) = f(\boldsymbol{x})^{\lambda} \tag{13}$$

whose product satisfies $f_l(\boldsymbol{x}) f_r(\boldsymbol{x}) = f(\boldsymbol{x}) = r$. In two dimensions, we obtain the amplitude

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) \approx \frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S \prod_{n=1}^{N+1} \left[\int \mathrm{d}\rho_n\, \Phi(\rho_n) \right] \times \frac{(r_a/r_b)^{1-2\lambda}}{2\pi\hbar\epsilon_b^s \rho_b/M} \prod_{n=2}^{N+1} \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}^2 \triangle x_n}{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}/M} \right] \exp\left\{ -\frac{1}{\hbar} A^N \right\} \tag{14}$$

with the action

$$A^N = \sum_{n=1}^{N+1} \left[\frac{M(\boldsymbol{x}_n - \boldsymbol{x}_{n-1})^2}{2\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^{\lambda}} - \mathrm{i}\frac{e}{c} \boldsymbol{A}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) - \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^{\lambda} \frac{(E - V_n)^2}{2Mc^2} + \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^{\lambda} \frac{Mc^2}{2} \right]. \tag{15}$$

Since the path integral represents the general relativistic resolvent operator, all results must be independent of the splitting parameter λ after going to the continuum limit. Choosing $\lambda = 1/2$, we obtain the continuum limit

$$A_E[\boldsymbol{x}, \boldsymbol{x}'] = \int \mathrm{d}s \left[\frac{M x'^2}{2\rho r} - \mathrm{i}\frac{e}{c} \boldsymbol{A} \cdot \boldsymbol{x}' - \rho r \frac{(E - V)^2}{2Mc^2} + \rho r \frac{Mc^2}{2} \right]. \tag{16}$$

We now solve the s-sliced ABC system as in the case of the two-dimensional Coulomb problem without the Aharonov–Bohm potential [2]. We introducing the *Levi-Civitá* transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u^1 & -u^2 \\ u^2 & u^1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \tag{17}$$

and write this in a matrix form:

$$\boldsymbol{x} = A(\boldsymbol{u})\boldsymbol{u}. \tag{18}$$

For every slice, the coordinate transformation reads

$$\boldsymbol{x}_n = A(\boldsymbol{u}_n)\boldsymbol{u}_n \tag{19}$$

yielding

$$(\triangle \boldsymbol{x}_n^i)^2 = 4\bar{u}_n^2 (\triangle \boldsymbol{u}_n^i)^2 \tag{20}$$

where $\bar{u}_n \equiv (\boldsymbol{u}_n + \boldsymbol{u}_{n-1})/2$. For the sliced AB potential, i.e.

$$\boldsymbol{A}_n = -2g \frac{(x_2)_n \hat{e}_1 - (x_1)_n \hat{e}_2}{r_n^2} \tag{21}$$

the *Levi-Civitá* transformation yields

$$\boldsymbol{A}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) = -2g \frac{(x_2)_n (\triangle x_1)_n - (x_1)_n (\triangle x_2)_n}{r_n^2} = -4g \frac{u_n^2 \triangle u_n^1 - u_n^1 \triangle u_n^2}{\boldsymbol{u}_n^2}. \tag{22}$$

Thus we obtain for the path integral of equation (14) the Duru–Kleinert-transformed expression:

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \frac{\mathrm{i}\hbar}{2Mc} \int_0^\infty \mathrm{d}S\, \mathrm{e}^{SEe^2/\hbar Mc^2} \frac{1}{4} [G(\boldsymbol{u}_b, \boldsymbol{u}_a; S) + G(-\boldsymbol{u}_b, \boldsymbol{u}_a; S)] \tag{23}$$

where $G(\boldsymbol{u}_b, \boldsymbol{u}_a; S)$ is the s-sliced amplitude of a harmonic oscillator in an Aharonov–Bohm vector potential corresponding to twice the magnetic field of equation (9) in $\boldsymbol{u}$-space:

$$G(\boldsymbol{u}_b, \boldsymbol{u}_a; S) = \prod_{n=1}^{N+1} \left[\int \mathrm{d}\rho_n \, \Phi(\rho_n) \right] \frac{1}{2\pi \bar{h} \epsilon_b^s \rho_b / M} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}^2 u_n}{2\pi \bar{h} \epsilon_n^s \rho_n / M} \right] \exp\left\{ -\frac{1}{\bar{h}} A^N \right\}$$

with the action

$$A^N = \sum_{n=1}^{N+1} \left\{ \frac{m(\triangle \boldsymbol{u}_n)^2}{2\epsilon_n^s \rho_n} - 2\mathrm{i}\frac{e}{c}(\boldsymbol{A}_n \cdot \triangle \boldsymbol{u}_n) + \epsilon_n^s \rho_n \frac{m\omega^2 \boldsymbol{u}_n^2}{2} - \epsilon_n^s \rho_n \frac{\bar{h}^2 4\alpha^2}{2m\boldsymbol{u}_n^2} \right\}. \tag{24}$$

Here

$$m = 4M \qquad \omega^2 = \frac{M^2 c^4 - E^2}{4M^2 c^2} \tag{25}$$

and

$$\boldsymbol{A}_n \cdot \triangle \boldsymbol{u}_n = -2g \frac{u_n^2 \triangle u_n^1 - u_n^1 \triangle u_n^2}{\boldsymbol{u}_n^2}. \tag{26}$$

The symmetrization in $\boldsymbol{u}_b$ in equation (23) is necessary since for each path from $\boldsymbol{x}_a$ to $\boldsymbol{x}_b$ there are two paths in the square root space, one from $\boldsymbol{u}_a$ to $\boldsymbol{u}_b$ and one from $\boldsymbol{u}_a$ to $-\boldsymbol{u}_b$.

As in the two-dimensional Coulomb problem, there are no s-slicing corrections [2].

Let us now analyse the effect of the magnetic interaction upon the Coulomb system, defining the azimuthal angle $\varphi(\boldsymbol{u}) = \arctan(u^2/u^1) = \theta(\boldsymbol{x})/2$ in the u-plane, so that $A_\mu = 2g\partial_\mu \varphi$, $B_3 = 2g\epsilon_{3jk}\partial_j \partial_k \varphi$. Note that the derivatives in front of $\varphi(\boldsymbol{u})$ commute everywhere, except at the origin where Stokes' theorem yields

$$\int \mathrm{d}^2 u (\partial_1 \partial_2 - \partial_2 \partial_1)\varphi = \oint \mathrm{d}\varphi = 2\pi. \tag{27}$$

The magnetic flux through the tube is defined by the integral

$$\Phi = \int \mathrm{d}^2 u \, B_3. \tag{28}$$

A comparison with the equation for $\varphi(\boldsymbol{u})$ shows that the coupling constant g is related to the magnetic flux by

$$g = \frac{\Phi}{4\pi}. \tag{29}$$

When inserting $A_\mu = 2g\partial_\mu \varphi$ into equation (24), the interaction takes the form

$$A_{\mathrm{mag}} = -2\bar{h}\mu_0 \int_0^S \mathrm{d}s \, \varphi'(s) \tag{30}$$

where $\varphi(s) \equiv \varphi(\boldsymbol{u}(s))$, and μ_0 is the dimensionless number

$$\mu_0 \equiv -\frac{2eg}{\bar{h}c}. \tag{31}$$

The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in spacetime can be considered as being closed at infinity, and the integral

$$n = \frac{1}{2\pi} \int_0^S \mathrm{d}s \, \varphi' \tag{32}$$

is the topological invariant with integer values of the winding number n. The magnetic interaction is therefore purely topological, its value being

$$A_{\mathrm{mag}} = -\bar{h}\mu_0 4\pi n. \tag{33}$$

After adding this to the action of equation (24) in the radial decomposition of the relativistic path integral [3, 6, 7], we rewrite the sum over the azimuthal quantum numbers k via Poisson's summation formula

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} \mathrm{d}\mu \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi n\mu \mathrm{i}} f(\mu) \tag{34}$$

and obtain

$$G(\boldsymbol{u}_b, \boldsymbol{u}_a; S) = \int_{-\infty}^{\infty} \mathrm{d}\mu \frac{1}{\sqrt{u_b u_a}} G(u_b, u_a; S)_\mu \times \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}(\mu-2\mu_0)(\varphi_b+2n\pi-\varphi_a)}. \tag{35}$$

Since the winding number n is often not easy to measure experimentally, let us extract observable consequences which are independent of n. The sum over all n forces μ to be equal to $2\mu_0$ modulo an arbitrary integer number. The result is

$$G(\boldsymbol{u}_b, \boldsymbol{u}_a; S) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{u_b u_a}} G(u_b, u_a; S)_{k+2\mu_0} \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}k(\varphi_b-\varphi_a)}. \tag{36}$$

We now choose the gauge $\rho(s) = 1$ in equation (24). This leads to the Duru–Kleinert transformed action

$$A^N = \int_0^S \mathrm{d}s \left[\frac{m\boldsymbol{u}'^2}{2} - 2\mathrm{i}\frac{e}{c}(\boldsymbol{A}\cdot\boldsymbol{u}') + \frac{m\omega^2\boldsymbol{u}^2}{2} - \frac{4\hbar^2\alpha^2}{2m\boldsymbol{u}^2} \right] \tag{37}$$

where α denotes the fine-structure constant $\alpha \equiv e^2/\hbar c \approx 1/137$. This action describes a particle of mass $m = 4M$ moving as a function of the 'pseudotime' s in an Aharonov–Bohm field and a harmonic oscillator potential of frequency

$$\omega^2 = \frac{M^2c^4 - E^2}{4M^2c^2}. \tag{38}$$

In addition, there is an extra attractive potential $V_{\mathrm{extra}} = -4\hbar^2\alpha^2/2m\boldsymbol{u}^2$ looking like an inverted centrifugal barrier which is conveniently parametrized with the help of a corresponding angular momentum l_{extra}, whose square is negative: $l^2_{\mathrm{extra}} \equiv -4\alpha^2$, writing $V_{\mathrm{extra}} = \hbar^2 l^2_{\mathrm{extra}}/2m\boldsymbol{u}^2$. Such an extra potential can easily be incorporated into the amplitude of the pure Coulomb system by a technique developed in the treatment of the radial part of the harmonic oscillator path integral†, yielding a radial amplitude for the azimuthal quantum number k:

$$G(u_b, u_a; S)_k = \frac{m}{\hbar} \frac{\omega\sqrt{u_b u_a}}{\sinh \omega s} \exp\left[-\frac{m\omega}{2\hbar}(u_b^2 + u_a^2)\coth \omega s \right] I_{\sqrt{|k|^2-4\alpha^2}}\left(\frac{m}{\hbar} \frac{\omega u_b u_a}{\sinh \omega s} \right) \tag{39}$$

where I_ν is the modified Bessel function. Also incorporating the effect of the Aharonov–Bohm potential yields

$$G(u_b, u_a; S)_{k+2\mu_0} = \frac{m}{\hbar} \frac{\omega\sqrt{u_b u_a}}{\sinh \omega s} \times \exp\left[-\frac{m\omega}{2\hbar}(u_b^2 + u_a^2)\coth \omega s \right] I_{\sqrt{|k+2\mu_0|^2-4\alpha^2}}\left(\frac{m}{\hbar} \frac{\omega u_b u_a}{\sinh \omega s} \right). \tag{40}$$

These radial amplitudes can now be combined with angular wavefunctions to find the full amplitude of equation (36).

† See sections 8.6 and 16.6 in [2]; and [12].

Inserting the result into the integral representation of equation (23) for the resolvent, we use polar coordinates in $\boldsymbol{x}$-space with $\theta = 2\varphi$, $r = u^2$, and obtain the expression

$$G(\boldsymbol{x}_b, \boldsymbol{x}_a; E) = \sum_{k=-\infty}^{\infty} G(r_b, r_a; E)_k \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}k(\theta_b - \theta_a)} \tag{41}$$

where

$$G(r_b, r_a; E)_k = \frac{\mathrm{i}\hbar}{2Mc}\frac{2M}{\hbar}\int_0^\infty \mathrm{d}S\, \mathrm{e}^{e^2 ES/\hbar Mc^2} \times \frac{\omega}{\sinh\omega s}\exp\left[-\frac{m\omega}{2\hbar}(r_b + r_a)\coth\omega s\right] I_{\sqrt{4|k+\mu_0|^2 - 4\alpha^2}}\left(\frac{m}{\hbar}\frac{\omega\sqrt{r_b r_a}}{\sinh\omega s}\right). \tag{42}$$

The integral can be calculated with the help of the formula

$$\int_0^\infty \mathrm{d}y\, \frac{\mathrm{e}^{2\nu y}}{\sinh y}\exp\left[-\frac{t}{2}(\zeta_a + \zeta_b)\coth y\right] I_\mu\left(\frac{t\sqrt{\zeta_b\zeta_a}}{\sinh y}\right) = \frac{\Gamma((1+\mu)/2 - \nu)}{t\sqrt{\zeta_b\zeta_a}\Gamma(\mu+1)} W_{\nu,\mu/2}(t\zeta_b) M_{\nu,\mu/2}(t\zeta_a) \tag{43}$$

with the range of validity

$$\zeta_b > \zeta_a > 0 \qquad \mathrm{Re}\,[(1+\mu)/2 - \nu] > 0$$
$$\mathrm{Re}\,(t) > 0 \qquad |\arg t| < \pi$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are the Whittaker functions [15, p 1087]. In this way, we obtain the final result for the radial amplitude valid for $u_b > u_a$:

$$\begin{aligned} G(r_b, r_a; E)_k = {} & \frac{\mathrm{i}\hbar}{2Mc}\frac{Mc}{\sqrt{M^2c^4 - E^2}} \\ & \times \frac{\Gamma(1/2 + \sqrt{|k+\mu_0|^2 - \alpha^2} - E\alpha/\sqrt{M^2c^4 - E^2})}{\sqrt{r_a r_b}\Gamma(2\sqrt{|k+\mu_0|^2 - \alpha^2} + 1)} \\ & \times W_{E\alpha/\sqrt{M^2c^4-E^2}, \sqrt{|k+\mu_0|^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{M^2c^4 - E^2}\, r_b\right) \\ & \times M_{E\alpha/\sqrt{M^2c^4-E^2}, \sqrt{|k+\mu_0|^2-\alpha^2}}\left(\frac{2}{\hbar c}\sqrt{M^2c^4 - E^2}\, r_a\right). \end{aligned} \tag{44}$$

The energy spectra and wavefunctions can be extracted from the poles of equation (44). For convenience, we define the following variables

$$\begin{aligned} \kappa &= \frac{1}{\hbar c}\sqrt{M^2c^4 - E^2} \\ \nu &= \frac{\alpha E}{\sqrt{M^2c^4 - E^2}} \\ \tilde{k} &= \sqrt{|k+\mu_0|^2 - \alpha^2} - 1/2. \end{aligned} \tag{45}$$

From the poles of $G(r_b, r_a; E)_k$, we find that the energy levels must satisfy the equality

$$-\nu + \tilde{k} + 1 = -n_r \qquad n_r = 0, 1, 2, 3, \ldots. \tag{46}$$

Expanding this equation into a power of α, we get

$$E_{nk} = \pm Mc^2\left\{1 - \frac{1}{2}\left[\frac{\alpha}{n_r + |k+\mu_0| + 1/2}\right]^2 - \frac{\alpha^4}{[n_r + |k+\mu_0| + 1/2]^3} \times \left[\frac{1}{2|k+\mu_0|} - \frac{3}{8[n_r + |k+\mu_0| + 1/2]}\right] + \cdots\right\}. \tag{47}$$

The relativistic energy spectra can also be obtained from the local approach using the time-independent Klein–Gordon equation. In the non-relativistic limit, the spectra reduces to that in [14, 15]. It is worth noting that if the flux is quantized, i.e. $4\pi g = 2\pi\hbar c/e \times$ integer, then $|k + \mu_0|$ is an integer and the spectrum is that of the relativistic hydrogen atom. In this case, there is no Aharonov–Bohm effect.

The pole positions, which satisfy $\nu = \tilde{n}_k \equiv n + \tilde{k} - |k|$ $(n = |k|+1, |k|+2, |k|+3, \ldots)$, correspond to the bound states of the two-dimensional relativistic ABC system. Near the positive-energy poles, we use the behaviour for $\nu \approx \tilde{n}_k$,

$$-\mathrm{i}\Gamma(-\nu + \tilde{k} + 1)\frac{M}{\hbar\kappa} \approx \frac{(-)^{n_r}}{\tilde{n}_k^2 n_r!}\frac{1}{\tilde{a}_H}\left(\frac{E}{Mc^2}\right)^2 \frac{\mathrm{i}\hbar 2Mc^2}{E^2 - E_{nk}^2} \tag{48}$$

with $\tilde{a}_H \equiv a_H Mc^2/E$ being the modified energy-dependent Bohr radius and $n_r = n - |k| - 1$ the radial quantum number, to extract the amplitude of the two-dimensional ABC system

$$\begin{aligned} G(r_b, r_a; E)_k &= \left(\frac{\hbar}{2Mc}\right)\frac{1}{\sqrt{r_a r_b}}\sum_{n=|k|+1}^{\infty}\left(\frac{E}{Mc^2}\right)^2\frac{\mathrm{i}\hbar 2Mc^2}{E^2 - E_{nk}^2} \\ &\quad\times\frac{1}{[(2\tilde{k}+1)!]^2}\frac{1}{\tilde{n}_k^2\tilde{a}_H}\frac{(\tilde{n}_k + \tilde{k})!}{(n - |k| - 1)!}\,\mathrm{e}^{-(r_b + r_a)/\tilde{a}_H\tilde{n}_k}\left(\frac{2r_b}{\tilde{a}_H\tilde{n}_k}\frac{2r_a}{\tilde{a}_H\tilde{n}_k}\right)^{\tilde{k}+1} \\ &\quad\times M\left(-n + |k| + 1, 2\tilde{k} + 2; \frac{2r_b}{\tilde{a}_H\tilde{n}_k}\right) M\left(-n + |k| + 1, 2\tilde{k} + 2; \frac{2r_a}{\tilde{a}_H\tilde{n}_k}\right) \\ &= \left(\frac{\hbar}{2Mc}\right)\frac{1}{\sqrt{r_a r_b}}\sum_{n=|k|+1}^{\infty}\left(\frac{E}{Mc^2}\right)^2\frac{\mathrm{i}\hbar 2Mc^2}{E^2 - E_{nk}^2} R_{nk}(r_b)R_{nk}^*(r_a) + \cdots \end{aligned} \tag{49}$$

where we have expressed the Whittaker function $M_{\lambda,\mu}(z)$ in terms of the Kummer functions $M(a, b; z)$ [13, p 1087]

$$M_{\lambda,\mu}(z) = z^{\mu+1/2}\,\mathrm{e}^{-z/2}M(\mu - \lambda + 1/2, 2\mu + 1; z). \tag{50}$$

From equation (49), we obtain the radial wavefunctions

$$\begin{aligned} R_{nk}(r) &= \frac{1}{(2\tilde{k}+1)!}\sqrt{\frac{(\tilde{n}_k + \tilde{k})!}{(n - |k| - 1)!}}\frac{1}{\tilde{n}_k\tilde{a}_H^{1/2}} \\ &\quad\times\left(\frac{2r}{\tilde{a}_H\tilde{n}_k}\right)^{\tilde{k}+1}\mathrm{e}^{-r/\tilde{a}_H\tilde{n}_k}M\left(-n + |k| + 1, 2\tilde{k} + 2; \frac{2r}{\tilde{a}_H\tilde{n}_k}\right). \end{aligned} \tag{51}$$

It could easily found that, when the vector potential vanishes, equation (51) is the same as the two-dimensional relativistic Coulomb wavefunction.

Before extracting the continuous wavefunction we note that the parameter κ is real for $|E| < Mc^2$. For $|E| > Mc^2$, the square root in equation (45) has two imaginary solutions

$$\kappa = \mp\mathrm{i}\varsigma \qquad \varsigma = \frac{1}{\hbar c}\sqrt{E^2 - M^2c^4} \tag{52}$$

corresponding to

$$\nu = \pm\mathrm{i}\tilde{\nu} \qquad \tilde{\nu} = \frac{E\alpha}{\hbar c\varsigma}. \tag{53}$$

Therefore the amplitude has a right-handed cut for $E > Mc^2$ and $E < -Mc^2$. For simplicity, we will only consider the positive energy cut.

The continuous wavefunction is recovered from the discontinuity of the amplitudes $G(r_b, r_a; E)_k$ across the cut in the complex E plane. Hence, we have

$$
\begin{aligned}
\operatorname{disc} G(r_b, r_a; E > Mc^2) &\equiv G(r_b, r_a; E + \mathrm{i}\eta)_k - G(r_b, r_a; E - \mathrm{i}\eta)_k \\
&= \frac{\hbar}{2Mc\sqrt{r_a r_b}} \frac{M}{\hbar\varsigma} \left[\frac{\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{k} + 1)}{(2\tilde{k} + 1)!} W_{\mathrm{i}\tilde{\nu}, \tilde{k}+1/2}(-2\mathrm{i}\varsigma r_b) M_{\mathrm{i}\tilde{\nu}, \tilde{k}+1/2}(-2\mathrm{i}\varsigma r_a) \right. \\
&\quad \left. + (\tilde{\nu} \to -\tilde{\nu}) \right]
\end{aligned} \tag{54}
$$

where the notation η is an infinitesimal parameter. Using the relations [16, p 299]

$$
M_{\kappa,\mu}(z) = \mathrm{e}^{\pm \mathrm{i}\pi(2\mu+1)/2} M_{-\kappa,\mu}(-z) \tag{55}
$$

where the sign is positive or negative in terms of $\operatorname{Im} z > 0$ or $\operatorname{Im} z < 0$, and [13, p 1090]

$$
\begin{aligned}
W_{\lambda,\mu}(z) &= \mathrm{e}^{\mathrm{i}\pi\lambda}\, \mathrm{e}^{-\mathrm{i}\pi(\mu+1/2)} \frac{\Gamma(\mu + \lambda + 1/2)}{\Gamma(2\mu + 1)} \\
&\quad \times \left[M_{\lambda,\mu}(z) - \frac{\Gamma(2\mu + 1)}{\Gamma(\mu - \lambda + 1/2)} \mathrm{e}^{-\mathrm{i}\pi\lambda} W_{-\lambda,\mu}(\mathrm{e}^{-\mathrm{i}\pi} z) \right]
\end{aligned} \tag{56}
$$

is valid only for $\arg(z) \in (-\pi/2, 3\pi/2)$ and $2\mu \neq -1, -2, -3, \ldots$. The discontinuity of the amplitude is found to be

$$
\begin{aligned}
\operatorname{disc} G(r_b, r_a; E > Mc^2) &= \frac{\hbar}{2Mc\sqrt{r_a r_b}} \frac{M}{\hbar\varsigma} \frac{|\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{k} + 1)|^2}{|\Gamma(2\tilde{k} + 2)|^2} \\
&\quad \times \mathrm{e}^{\pi\tilde{\nu}} M_{-\mathrm{i}\tilde{\nu}, \tilde{k}+1/2}(2\mathrm{i}\varsigma r_b) M_{\mathrm{i}\tilde{\nu}, \tilde{k}+1/2}(-2\mathrm{i}\varsigma r_a).
\end{aligned} \tag{57}
$$

Thus we have

$$
\begin{aligned}
&\int_{Mc^2}^{\infty} \frac{\mathrm{d}E}{2\pi\hbar} \operatorname{disc} G(r_b, r_a; E > Mc^2) \\
&\quad = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \frac{(\hbar c)^2 \varsigma\, \mathrm{d}\varsigma}{\sqrt{M^2c^4 + (\hbar c\varsigma)^2}} \operatorname{disc} G(r_b, r_a; E > Mc^2) \\
&\quad = \frac{\hbar}{2Mc\sqrt{r_a r_b}} \int_{-\infty}^{\infty} \mathrm{d}\varsigma \left(\frac{E}{Mc^2} \right) R_{\varsigma k}(r_b) R^*_{\varsigma k}(r_a).
\end{aligned} \tag{58}
$$

From equation (58), we obtain the continuous radial wavefunction of the two-dimensional relativistic ABC system

$$
R_{\varsigma k}(r) = \sqrt{\frac{1}{2\pi}} \frac{1}{[1 + (c\hbar\varsigma/Mc^2)^2]^{1/2}} \frac{|\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{k} + 1)|}{(2\tilde{k} + 1)!} \mathrm{e}^{\pi\tilde{\nu}/2} M_{\mathrm{i}\tilde{\nu}, \tilde{k}+1/2}(-2\mathrm{i}\varsigma r) \tag{59}
$$

$$
\begin{aligned}
&= \sqrt{\frac{1}{2\pi}} \frac{1}{[1 + (c\hbar\varsigma/Mc^2)^2]^{1/2}} \frac{|\Gamma(-\mathrm{i}\tilde{\nu} + \tilde{k} + 1)|}{(2\tilde{k} + 1)!} \\
&\quad \times \mathrm{e}^{\pi\tilde{\nu}/2}\, \mathrm{e}^{\mathrm{i}\varsigma r} (-2\mathrm{i}\varsigma r)^{\tilde{k}+1} \times M(-\mathrm{i}\tilde{\nu} + \tilde{k} + 1, 2\tilde{k} + 2; -2\mathrm{i}\varsigma r).
\end{aligned} \tag{60}
$$

3. Conclusions

In this paper, Kleinert's relativistic path integral with the magnetic interaction is studied. As an application, we have calculated the path integral of the relativistic ABC system in two dimensions. The ABC case serves as a prototype of path integral for arbitrary relativistic potential systems.

It is our hope that our studies would help to achieve the ultimate goal of obtaining a comprehensive and complete path integral description of quantum mechanics and quantum field theory, including quantum gravity and cosmology.

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